Main Theorem of [Litt, 2018]

Theorem (Litt '18)

Let X/k be a normal, geom. conn. variety, where $k/\mathbb{Q}$ fin. gen. field ext.. Let ℓ be a prime.
Then there ex. $N = N(X, \ell) > 0$ s.t. any arithmetic repr.
$$\rho : \pi_1^{et}(X_{\bar{k}}) \longrightarrow GL_n(\mathbb{Z}_\ell)$$
which is trivial mod ℓ^N is unipotent.

ρ arithm. $:\Leftrightarrow$ $\exists\, k'/k$ finite, $\exists\, \tilde{\rho} : \pi_1^{et}(X_{k'}) \to GL_n(\mathbb{Z}_\ell)$ s.t. ρ is a subquot. of $\tilde{\rho}|_{\pi_1^{et}(X_{\bar{k}})}$

Proof:

1) Reduction to X = affine curve

- $X \hookrightarrow \tilde{X} \overset{f}{\dashrightarrow} \bar{P}$
  - $\tilde{X}$: sm. cptf.
  - f: birat. map from Chow's lemma
  - $\bar{P}$: projective

$\rightsquigarrow$ Restrict to $U \subset X$ open, smooth $\Rightarrow$ f morph. and
$$\pi_1(U) := \pi_1^{et}(U) \twoheadrightarrow \pi_1(X)$$

$\Rightarrow$ may assume X q-proj., smooth.
Use "Lefschetz hyperplane theorem"-like argument [Deligne]:
$\exists$ smooth curve C/k, $C \longrightarrow X$ s.t. $\pi_1(C_{\bar{k}}, \bar{x}) \twoheadrightarrow \pi_1(X_{\bar{k}}, \bar{x})$, $\forall \bar{x}$ geom. pt. of C.
$\rightsquigarrow$ Restrict to $V \subset C$ open, may assume:

$X := C$ affine, smooth curve

- Then $\pi_1^\ell(X_{\bar{k}}, \bar{x})$ is fin. gen. free pro-ℓ group

  ($X_{\bar{k}}$ = smooth, proj. curve of genus g with n points removed
  $\Rightarrow \pi_1^{top}(X_{\mathbb{C}}^{an}, \bar{x})$ = free grp in $2g+n-1$ generators)

  Therefore

  example in talk 2

  $$\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \cong \mathbb{Z}_\ell \langle\langle T_1, \dots, T_s \rangle\rangle$$ ← non-commuting power series

  and

  (*) $\forall n \geq 0$: I^n / I^{n+1} is $\mathbb{Z}_\ell$-torsion free (I = augm. ideal)

  is satisfied.

- May assume: $\bar{x}$ comes from rat. pt. of X ($k \rightsquigarrow k'/k$ fin.)

  $\Rightarrow$ Get section $\pi_1(X_k, \bar{x}) \hookleftarrow G_k$

  $\Rightarrow$ Get G_k-action on $\pi_1^\ell(X_{\bar{k}}, \bar{x})$ by autom.

2) <u>Reductions for ϱ</u>

Consider $\varrho: \pi_1(X_{\bar{k}}, \bar{x}) \longrightarrow GL_n(\mathbb{Z}_\ell)$, trivial mod ℓ^N

(N to be determined later)

Have socle filtration for $\varrho \otimes \mathbb{Q}_\ell$

$$0 = V_0 \subset V_1 \subset \dots \subset \varrho \otimes \mathbb{Q}_\ell$$

$\rightsquigarrow$ Replace ϱ by $(\varrho \cap V_i) / (\varrho \cap V_{i-1})$

$\Rightarrow$ Enough to consider semi-simple $\varrho \otimes \mathbb{Q}_\ell = \bigoplus_j W_j$ (irred.)

$\Rightarrow$ —— " —— $\varrho \otimes \mathbb{Q}_\ell$ irred.

Lemma:

Suppose ρ like in Thm. and $\rho \otimes \mathbb{Q}_\ell$ irred.

Then, there ex. k'/k fin. and repr. $\beta : \pi_1(X_{k'}, \bar{x}) \to GL_n(\mathbb{Z}_\ell)$ s.t.

- ρ is a subquot. of $\beta|_{\pi_1(X_{\bar{k}}, \bar{x})}$ and
- $\beta|_{\pi_1(X_{\bar{k}}, \bar{x})}$ triv. mod ℓ^ν.

Idea of proof:

ρ arithm. $\Rightarrow$ ρ comes from $\gamma : \pi_1(X_{k'}) \to GL_n(\mathbb{Z}_\ell)$

$\rightsquigarrow$ use socle arguments on γ to construct β. $\square$

$\Rightarrow$ Suffices to show: Such $\beta|_{\pi_1(X_{\bar{k}}, \bar{x})}$ is unipotent.

3) Group ring machinery

$$\begin{array}{ccc}
\pi_1(X_{\bar{k}}, \bar{x}) & \xrightarrow{\beta|_{\pi_1(X_{\bar{k}}, \bar{x})}} & GL_n(\mathbb{Z}_\ell) \\
\downarrow & \overset{\exists \beta'}{\nearrow} & \downarrow \\
\pi_1^\ell(X_{\bar{k}}, \bar{x}) & & GL_n(\mathbb{Z}_\ell/\ell^\nu \mathbb{Z}_\ell)
\end{array}$$

$\longleftarrow$ Ker is pro-ℓ group

Had equivalence:

$\{$f.g. free cont. $\mathbb{Z}_\ell$-repr. of $\pi_1^\ell\}$ $\longleftrightarrow$ $\{\mathbb{Z}_\ell[[\pi_1^\ell]]$-mod., fin. gen. free over $\mathbb{Z}_\ell\}$.

Set

$$\begin{array}{ccc}
\mathbb{Z}_\ell[[\pi_1^\ell]] & \xrightarrow{\mathbb{Z}_\ell[[\beta']]} & M_n(\mathbb{Z}_\ell) \\
\downarrow & \circlearrowleft & \downarrow \\
\mathbb{Q}_\ell[[\pi_1^\ell]]^{\leq \ell^{-r}} & \xrightarrow{\hat{\beta}} & M_n(\mathbb{Q}_\ell)
\end{array}$$

injectivity uses $(*)$ $\longrightarrow$ (left vertical arrow)

$(\square)$

$\exists \hat{\beta}$ cont. ring. homom. w.r.t. r-Gauss-norm $\forall\, 0 < r < \nu$

$\nwarrow$ from last talk

• Had thm. in talk 4:

$\forall \alpha \in \mathbb{Z}_\ell$ close to 1 $\exists\, \sigma_\alpha \in G_\xi$:

$\forall i \geq 0$: σ_α acts on $gr_W^{-i}\, \mathbb{Q}_\ell[\![\pi_1^\ell]\!]$ via $\alpha^i \cdot id$.

Fix such $\alpha \in \mathbb{Z}_\ell^\times$, α not root of unity, and such $\sigma_\alpha \in G_\xi$.

• By thm last time:

$\exists\, r_\alpha > 0\ \forall r > r_\alpha$:

(i) σ_α-action on $\mathbb{Q}_\ell[\![\pi_1^\ell]\!]^{\leq \ell^{-r}}$ admits a set of σ_α-eigenvectors with dense span.

(ii) $\forall i \geq 0$: σ_α-action on $W^{-i}\mathbb{Q}_\ell[\![\pi_1^\ell]\!]^{\leq \ell^{-r}}$ —— " —— and eigenvalues $\{\alpha^i, \alpha^{i+1}, \dots\}$.

⇝ Let $N :=$ least strictly greater integer than r_α. Choose $r_\alpha < r < N$.

• Let $m \in \mathbb{N}$ s.t. $\sigma_\alpha^m \in G_{\xi'}$. ($\xi'/\xi$ from lemma)

Have homom.

$$G_{\xi'} \xrightarrow{s} \pi_1(X_{\bar{k}}, \bar{x}) \xrightarrow{\beta} GL_n(\mathbb{Z}_\ell)$$

← use here that β already is def. on $\pi_1(X_{\bar{k}}, \bar{x})$

⇝ get σ_α^m-action on $M_n(\mathbb{Z}_\ell)$ by conj. with $(\beta \circ s)(\sigma_\alpha^m)$. This way (□) is σ_α^m-equivariant.

In part: $\tilde{\beta}$ sends σ_α^m-eigenvectors to σ_α^m-eigenvectors

• $\dim_{\mathbb{Q}_\ell} M_n(\mathbb{Q}_\ell) < \infty$

⇒ For $s \gg 0$, α^s is not eigenvalue of σ_α^m on $M_n(\mathbb{Q}_\ell)$

• (ii) ⇒ $\forall i \geq 0$: σ_α^m-action on $W^{-i}\mathbb{Q}_\ell[\![\pi_1^\ell]\!]^{\leq \ell^{-r}}$ admits eigenvectors with dense span and eigenvalues $\{\alpha^{m(i+j)} \mid j \geq 0\}$.

⇒ For $i \gg 0$ (s.t. $mi \geq s$): $\tilde{\beta}(W^{-i}\mathbb{Q}_\ell[\![\pi_1^\ell]\!]^{\leq \ell^{-r}}) = 0$

$\bigcup$

$W^{-i}\mathbb{Z}_\ell[\![\pi_1^\ell]\!]$

- In talk 4: $\mathfrak{W}^\bullet$-adic top $= I^\bullet$-adic top. on $\mathbb{Z}_\ell[[\pi_1^\ell]]$.

$\Rightarrow \exists t \geq 0$: $\hat{\beta}(I^t) = 0$

$\Rightarrow \forall g \in \pi_1(X_{\bar{k}}, \bar{x})$: $\left(\beta|_{\pi_1^{et}(X_{\bar{s}}, \bar{x})}(g) - \mathbb{1}_n\right)^t = \hat{\beta}(\underbrace{g-1}_{\in I = \ker(g \mapsto 1)})^t = \hat{\beta}((g-1)^t) = 0$

$\Rightarrow \beta|_{\pi_1^{et}(X_{\bar{s}}, \bar{x})}(g)$ unipotent.

$\square$

Examples

X affine curve

- Can replace $\mathrm{char}(k) = 0$ by assumption

  (†) $\ell \neq \mathrm{char}(k)$ and the image of $G_k \longrightarrow GL(H^1_{et}(X_{\bar{k}}, \mathbb{Z}_\ell))$ is open in its Zariski closure.

  Moreover, N only depends on the index thereof.
  If $\ell > 2$ and this index $= 1$, may take $N = 1$.

- Consider $\mathbb{P}^1_{\bar{k}} \setminus \{x_1, \dots, x_n\}$ ← distinct, and $K = k_0\left(\frac{x_a - x_b}{x_c - x_d} \,\middle|\, 1 \leq a < b < c < d \leq n\right)$ ← prime subfield of k

  $\rightsquigarrow$ $\mathbb{P}^1_{\bar{k}} \setminus \{x_1, \dots, x_n\}$ admits model X over K and

  index of $\mathrm{Im}(G_K \longrightarrow GL(H^1_{et}(X_{\bar{K}}, \mathbb{Z}_\ell)))$

  $=$ index of $\mathrm{Im}(G_K \xrightarrow{\chi} \mathbb{Z}_\ell^\times)$ ← cyclotomic char.

  In part: If $\ell > 2$ and χ surj., then any non-triv. geom. repr. of $\pi_1^{et}(\mathbb{P}^1_{\bar{k}} \setminus \{x_1, \dots, x_n\})$ must be non-triv. mod ℓ.

  ("geom" + "$\mathrm{char}(k) = 0$" $\Rightarrow$ "ss"
  "ss" + "unip." $\Rightarrow$ "triv")

(1) $\ell = 3$ or 5, $Y(\ell)$ = mod. curve param. ell. curves with full level ℓ structure

Have univ. family $E \longrightarrow Y(\ell)$, $\bar{x}$ geom. pt. of $Y(\ell)$

$\rightsquigarrow$ taut. monodromy repr. $\rho : \pi_1^{et}(Y(\ell), \bar{x}) \longrightarrow GL(T_\ell(E_{\bar{x}}))$

ρ triv. mod ℓ, but not triv. itself!

$T_\ell(E_{\bar{x}}) = H^1(E_{\bar{x}}, \mathbb{Z}_\ell)^\vee$ ← $\mathbb{Z}_\ell$-dual

<u>But:</u> $K = \mathbb{Q}(\text{cross-ratios of cusps of } Y(\ell)) = \mathbb{Q}(\zeta_\ell)$

and $\chi : G_K \longrightarrow \mathbb{Z}_\ell^\times$ is not surj. (factors through $1 + \ell\mathbb{Z}_\ell$)

(2) Again $Y(3)$

$\rightsquigarrow Y(3)_{\mathbb{C}} \cong \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty, \lambda\}$, where $\lambda \in \mathbb{Q}(\zeta_3)$

Let

$X := \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty, \beta\}$, for $\beta \in \mathbb{Q} \setminus \{0,1\}$

$\Rightarrow$ Have homeom. $X(\mathbb{C})^{an} \xrightarrow{\dot{\jmath}} Y(3)(\mathbb{C})^{an}$

monodromy repr.

Then

$\tilde{\varrho}: \pi_1^{top}(X(\mathbb{C})^{an}, j^{-1}(x)) \xrightarrow{j_*} \pi_1^{top}(Y(3)(\mathbb{C})^{an}, x) \xrightarrow{\varrho} GL(H^1(E_x(\mathbb{C})^{an}, \mathbb{Z}))$

triv. mod 3.

$\Rightarrow$ The induced $\hat{\varrho}: \pi_1^{et}(X, j^{-1}(x)) \longrightarrow GL(H^1(E_x, \mathbb{Z}_\ell))$

is not geom.

(3) $X/\mathbb{Q}$ smooth, proper curve of genus 2.

Recall: $\pi_1(X(\mathbb{C})^{an}) = \langle a_1, b_1, a_2, b_2 \rangle / ([a_1, b_1][a_2, b_2] = 1)$

Let ℓ prime, $A, B \in GL_n(\mathbb{Z}_\ell)$ non-unipotent s.t. $A, B \equiv 1 \bmod \ell^\nu$.

$\Rightarrow$ For $\nu \gg 0$, the repr.

$a_1 \mapsto A,\ b_1 \mapsto B,\ a_2 \mapsto B,\ b_2 \to A$

does not come from geom. and the induced

$\pi_1^{et}(X_{\bar{\mathbb{Q}}}) \longrightarrow GL_n(\mathbb{Z}_\ell)$

is not arithm..

Open questions:

Can one replace (†)?

Q: X/k smooth curve, $|k| < \infty$, ℓ prime, $\ell \neq char(k)$.

Is there $r = r(X)$ s.t. $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}$ admits Frob.-eigenvectors with dense span?